Concursul interjudețean de matematică
’’Memorialul Alexandru Cojocaru’’
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Clasa a XII-a

1. Fie I(a) = $\int_0^1 \frac{arctgx}{x^2+x+a}$ dx , a >o.

a) Sa se arate ca I(1) ≤ $\frac{p}{3\sqrt{3}}(\frac{p}{4}-\ln\sqrt{2})$.

b) Sa se calculeze I(2).

Prof. Marian Ursarescu, Roman

Dem.

a) f (x) = arctgx strict crescator ; g(x)=$\frac{1}{x^2+x+1}$ strcit descrectator; din inegalitatea Cebişev

$$\Rightarrow \int_0^1 \frac{arctgx}{x^2+x+1}dx \le \int_0^1 arctgxdx \cdot \int_0^1 \frac{1}{x^2+x+1}dx \qquad (1p)$$

$$\int_0^1 arctgxdx = \frac{p}{4}-\ln\sqrt{2} \qquad \int_0^1 \frac{1}{x^2+x+1}dx = \frac{p}{3\sqrt{3}} \qquad (1p)$$

$$\Rightarrow I(1) \le \frac{p}{3\sqrt{3}}\left(\frac{p}{4}-\ln\sqrt{2}\right) \qquad (1p)$$

b) I(2)=$\int_0^1 \frac{arctgx}{x^2+x+2}dx$ x=$\frac{1-t}{1+t} \Rightarrow dx = \frac{-2}{(1+t)^2}dt$ x=0 $\Rightarrow$t=0. (1p)

$$I(2)=\int_1^0 \frac{arctg\ \frac{1-}{1+x}}{\frac{(1-t)^2}{(1+t)^2}+\frac{1-t}{1+t}+2}\left(-\frac{2}{1+t^2}\right)dt = \int_0^1 \frac{arctg\ \frac{1-x}{1+x}}{t^2+t+2}dt \qquad (1p).$$

Dar arctg$\frac{1-x}{1+x}+arctgx=\frac{p}{4}$, $\forall$ x>-1 (1p)

$$I(2)=\int_0^1 \frac{\frac{p}{4}-arctgt}{t^2+t+2}dt = \frac{p}{4}\int_0^1 \frac{1}{t^2+t+2}dt - I(2)$$

$$\Rightarrow I(2) = \frac{p}{8}\int_0^1 \frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{t}}{2}\right)^2}dt = \frac{p}{8}\cdot\frac{2}{\sqrt{7}}\cdot arctg\frac{t+\frac{1}{2}}{\frac{\sqrt{t}}{2}}\Big|_0^1 \qquad (1p).$$

2. Fie $f : R \to R$ periodica , marginaita si astfel incat exista $x_0 \in R$ pentru care $l_s(x_0)$, $l_d(x_0)$ exista , sunt finite si distrincte. Determinati $a\in R$ penrtu care nu exista $\lim_{x\to\infty}\int_0^x (f(t)+a)dt$.

prof. Paul Georgescu si prof. Gabriel Popa, Iasi.

Solutie : Fie $T\in R_T^*$ o perioada a lui $\int$ si notam $F(x)=\int_0^x (f(t)+a)dt$, atunci

$$F(x+(n+1)T)-F(x+nT)=\int_{x+nT}^{x+(n+1)T}(f(t)+a)dt=\int_0^T f(t)dt+aT \Rightarrow F(x+nT)=F(x)+n\left(\int_0^T f(t)dt+aT\right) \forall$$

$n\in N^*$, $\forall$ $x\in[0,T)$ (1p). Cum f este marginita, rezulta ca F este marginita pe [0,T], iar dacva $\int_0^T f(t)dt+aT\neq 0$, rezulta imediat ca $\lim_{x\to\infty}F(x)=(+\infty)\cdot\operatorname{sgn}\left(\int_0^T f(t)dt+aT\right)$. (1p)

Fie acum $a=-\frac{1}{T}\cdot\int_0^T f(t)dt$ atunci $F(x+T)=\int_0^{x+t}(f(t)+a)dt=F9x)$, $\forall$ $x\in R$ (1p) , deci F este periodica de perioada T. Presupunem prin absurd ca F este constanta . Atunci $F(x_1)=F(x_2)$, $\forall$ $x_1,x_2\in R$, deci $\int_{x_1}^{x_2}(f(t)+a)dt=0$, $\forall$ x1,$x_2\in R$. Pentru $\varepsilon>0$ fixat, putem gasi $\delta>0$ astfel ca $(l_s-\delta+a)(x_2-x_1)\leq\int_{x_1}^{x_2}(f(t)+a)dt\leq(l_s+e+a)\cdot(x_2-x_1)$ pentru $x_0-\delta\leq x_1<x_2<x_0$, deci $l_s-\varepsilon+a\leq 0\leq l_s+\varepsilon+a$. Cum ε era arbitrar , obtinem ca $l_s(x_0)+a=0$ (2p). Analog , $l_d(x_0)+a=0$, deci $l_s(x_0)=l_d(x_0)=-a$, contradictie cu ipoteza (1p). Ramaneca F este periodica si neconstanta si deci nu are limita la $+\infty$ in cazul in care $\int_0^T f(t)dt+aT\neq 0$ (1p).

3. Sa se rezolve in Z_{21} ecuatia :

$$x^{12}+x^7+\hat{7}x^6+\widehat{14}x^4+\widehat{14}x^3+\hat{7}x=\hat{0} \qquad (1)$$

prof. Gabriel Popa , Iasi.

Solutie : Evident ca $x_1=\hat{0}$ este solutie a ecuatiei . Cautam in continuare $\alpha\in\{1,2,3,\ldots,20]$ a.i $\hat{a}\in Z_{21}$, sa fie solutie a ecuatiei. Distingem situatiile :
(0,5p)

(i) $(\alpha,21)=1$. Conform teoremei lui Euler , avem atunci ca $a^{j(21)}\equiv 1(\bmod 21)$ Cum $j(21)=j(3)\cdot j(7)=2\cdot 6=12$, iar $\hat{a}$ este solutie a ecuatiei (1), obtinem in Z_{21} egalitatea : $\hat{a}^7+\hat{7}\hat{a}^6+\widehat{14}\hat{a}^4+\widehat{14}\hat{a}^3+\hat{7}\hat{a}+\hat{1}=\hat{0}\Leftrightarrow\left(\hat{a}+\hat{1}\right)^7=\hat{0}\Leftrightarrow(3)\Leftrightarrow\hat{a}+\hat{1}=\hat{0}\Leftrightarrow\hat{a}=\widehat{20}$. Echivalenta (2) provine din formula binomului Newton, precum si din urmatoarele observatii : $\hat{C}_7^2=\widehat{21}=\hat{0}$; $\hat{C}_7^3=\widehat{35}=14$

Pentru (3) vom dovedi ca : $\hat{u}^7=\hat{0}\Leftrightarrow\hat{u}=\hat{0}$

Implicatia $''\Leftarrow''$ este evidenta. Pentru $''\Rightarrow''$, observam ca in ipoteza $\hat{u}^7=\hat{0}$, $\hat{u}$ nu poate fi nici inversabil in Z_{21}, de asemenea u nu poate fi nici par(in Z) ; ramane ca u $\in\{0,3,7,9,15\}$. Insa $\hat{3}^7=\hat{3},\hat{9}^7=\hat{9},\hat{7}^7=\hat{7},\widehat{15}^7=\widehat{15}$, dupa cum se poate verifica imediat. Deci $\hat{u}=\hat{0}$ (2,5p)

(ii) $\alpha\in\{7,14\}$. Cum $\hat{a}$ solutie pentru (1), avem :

$\hat{a}^{12}+\hat{a}^7+\hat{7}\hat{a}^6+\widehat{14}\hat{a}^4+\widehat{14}\hat{a}^3+\hat{7}\hat{a}=\hat{0}\Leftrightarrow\hat{a}^{12}+\hat{a}^7+\hat{7}\left(\hat{a}^6-\hat{a}^4\right)+\hat{7}\left(\hat{a}^3-\hat{a}\right)=\hat{0}\Leftrightarrow$

$\hat{a}^{12}+\hat{a}^7+\hat{7}\hat{a}^3\left(\hat{a}^3+\hat{a}\right)+\hat{7}(\hat{a}^3-\hat{a})=\hat{0}\Leftrightarrow(4)\Leftrightarrow\hat{a}^{12}+\hat{a}^7=\hat{0}\Leftrightarrow\hat{a}^7\left(\hat{a}^5+\hat{1}\right)=\hat{0}\Leftrightarrow(5)\Leftrightarrow$ Deoar

$a^5+1\in M_3\Leftrightarrow a=M_3-1\Leftrightarrow(ii)\Leftrightarrow\hat{a}=\widehat{14}$

ece $3\,|\,\hat{a}-a$, $\forall\alpha\in$ Z, atunci avem ca $\hat{7}\left(\hat{a}^3-\hat{a}\right)=\hat{0}$, ceea ce justifica (4). In plus , fiindca ipoteza (ii) arata ca $\alpha\in M_7$, avem imediat (5).

(iii) $a\in\{\hat{3},\hat{6},\hat{9},\widehat{12},\widehat{15},\widehat{18}\}$ Cum $\hat{a}$ solutie , avem:

Observam ca : a) $\alpha=M_7+1\Rightarrow\alpha^5+1=(M_7+1)^5+1=M_7+2$

b) $\alpha=M_7-1\Rightarrow\alpha^5+1=(M_7-1)^5+1=M_7$

c) $\alpha=M_7+1\Rightarrow\alpha^5+1=(M_7+2)^5+1=M_7+5$

d) $\alpha=M_7-2\Rightarrow\alpha^5+1=(M_7-2)^5+1=M_7-2$

e) $\alpha=M_7+3\Rightarrow\alpha^5+1=(M_7+3)^5+1=M_7+6$

f) $\alpha=M_7-3\Rightarrow\alpha^5+1=(M_7-3)^5+1=M_7-1$

rezulta ca in mod necesar $\alpha=M_7-1$, deci $\alpha=6$. pentru $\hat{a}=\hat{6}$, se vede ca: $\hat{a}^7\left(\hat{a}^5+\hat{1}\right)=\hat{a}^7\left(\hat{a}+\hat{1}\right)\left(\hat{a}^4-\hat{a}^3+\hat{a}^2-\hat{a}+\hat{1}\right)=\hat{0}\left(inZ_{21}\right)$ (2p)

asadar solutiile ecuatiei sunt:

$x_1=\hat{0},x_2=\hat{6},x_3=\widehat{14},x_4=\widehat{20}$