Concursul interjudețean de matematică
’’Memorialul Alexandru Cojocaru’’
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Clasa a XI-a

1. Fie A, B, $A^{-1}\in M_2(Z)$ si multimea $M=\{k\in Z/ (A+KB)^{-1}\in M_2(Z)\}$. Se se arate ca cardinalul $M\neq 2006$.

Prof. Gabriel Constantin, Roman

Solutie :

Pt. X, $x^{-1}\in M_2(Z) \Rightarrow \det x=\pm 1 \quad \Rightarrow \det (A+KB)=\pm 1$ 2p.

$\det(A+KB)=K^2\det B + \alpha K + \det A= f(k)$

daca $\exists$ k_1, k_2, k_3, k_4 a.i. $(A+K_i B)^{-1}\in M_2(Z)$, $i=\overline{1,4}$ atunci $f(k_1), f(k_2), f(k_3), f(k_4), f(0)\in \{\pm 1\} \Rightarrow$ f ia valoarea 1 sau –1 pentru 3 valori distincte$\Rightarrow$f= constant 3p.$\Rightarrow\exists$ o infinitate de valori K pentru care $(A+KB)^{-1}\in M_2(Z)\Rightarrow$cardinalul $M\neq 2006$ 2p.

Nota ! Pentru demonstatie pentru A si B particulare. 2p.

2. Fie $A\in M_3(R)$ inversabila a.i. $trA=trA^2=0$.
Sa se arate ca $\det(A+A^{-1})=\det A+\det A^{-1}$

Prof. Marian Ursarescu, Roman

Solutie : $\det(A+A^{-1})=\det(A^{-1}(A^2+I_3))=\det A^{-1}\det(A^2+I_3)=\frac{\det(A^2+I_3)}{\det A}$ (1)

Fie $p_a(x)=x^3-trA\,x^2+trA^A\,x-\det A$

$trA=0$ $trA^A=\frac{1}{2}((tr\,A)^2-tr\,A^2)=0\Rightarrow p_A(x)=x^3-\det A$ au valorile proprii λ_1, λ_2, λ_3. Fie $f(x)=x^2+1$. 1p.

$\det(A^2+I_3)=\det f(A)=f(\lambda_1)\,f(\lambda_2)\,f(\lambda_3)=(\lambda_1^2+1)(\lambda_2^2+1)(\lambda_3^2+1)$ 1p.

$p_A(x)=(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$

$p_i(x)=(i-\lambda_1)(i-\lambda_2)(i-\lambda_3)$

$p_{-i}(x)=(-i-\lambda_1)(-i-\lambda_2)(-i-\lambda_3)$ 2p.

$p_A(i)\,p_A(-i)=(i^2-\lambda_1^2)(i^2-\lambda_2^2)(i^2-\lambda_3^2)=(1+\lambda_1^2)(1+\lambda_2^2)(1+\lambda_3^2)$

$\det(A^2+I_3)=(-i-\det A)(i-\det A)=-(i+\det A)(i-\det A)=-(i^2-(\det A)^2)=1+(\det A)^2$ (2) 1p.

Din (1) si (2)$\Rightarrow\det(A+A^{-1})=\frac{1+(\det A)^2}{\det A}=\det A+\frac{1}{\det A}\Rightarrow\det(A+A^{-1})=\det A+\det A^{-1}$ 1p.

3. Fie sirurile $(a_n)_{n\geq 0,}$ $(b_n)_{n\geq 0,}$ $(c_n)_{n\geq 0,}$ definita astfel: a_0, b_0, c_0 >0 si

$$\sqrt{a_{n+1}+a_{n+1}b_n}+\sqrt{a_{n+1}+a_{n+1}c_n}=\sqrt{b_n+b_nc_n}+\sqrt{a_{n+1}+a_{n+1}b_n}$$
$$\sqrt{b_{n+1}+b_{n+1}c_n}+\sqrt{b_{n+1}+b_{n+1}a_n}=\sqrt{c_n+c_na_n}+\sqrt{a_n+a_nc_n}$$
$$\sqrt{c_{n+1}+c_{n+1}a_n}+\sqrt{c_{n+1}+c_{n+1}b_n}=\sqrt{a_n+a_nb_n}+\sqrt{b_n+b_na_n}$$

Sa se arate ca :

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty} b_n=\lim_{n\to\infty} c_n=tg^2\,\frac{arctg\sqrt{a_0}+arctg\sqrt{b_0}+arctg\sqrt{c_0}}{3}$$

prof. Gabriel Necula, Plopeni

Solutie: Prin inductie dupa n≥1, din conditia de existenta asupra primului radical, in fiecare relatie $\Rightarrow a_n\geq 0, b_n\geq 0, c_n\geq 0, \forall n\in N^*$. Pentru orice $a_n, b_n, c_n\geq 0$ exista $x_n, y_n, z_n\in[o, \frac{p}{2})$ a.i.

tg $x_n=\sqrt{a_n}$, tg $y_n=\sqrt{b_n}$, tg $z_n=\sqrt{c_n}$ 3p.

Din prima relatie se obtine

$$\sqrt{tg^2x_{n+1}+tg^2x_{n+1}tg^2y_n}+\sqrt{tg^2x_{n+1}+tg^2x_{n+1}tg^2z_n}=\sqrt{tg^2y_n+tg^2y_ntg^2z_n}+\sqrt{tg^2z_n+tg^2z_ntg^2y_n}\Rightarrow$$

$\Rightarrow$tg $x_{n+1}(\frac{1}{\cos y_n}+\frac{1}{\cos z_n})$=tg $z_n\,\frac{1}{\cos z_n}$ + tg $z_n\,\frac{1}{\cos y_n}$ de unde tg x_{n+1}=tg $\frac{y_n+z_n}{2}$.

Analog tg y_{n+1}=tg $\frac{z_n+x_n}{2}$ si tg z_{n+1} = tg $\frac{y_n+z_n}{2}$.

Rezulta $x_{n+1}=\frac{y_n+z_n}{2}$, $y_{n+1}=\frac{z_n+x_n}{2}$, $z_{n+1}=\frac{y_n+z_n}{2}$ (1)

Adunand $x_{n+1}+y_{n+1}+z_{n+1}=x_n+y_n+z_n$ $\forall n\in N$ de unde

$x_n+y_n+z_n=x_0+y_0+z_0=\alpha\in R, \forall n\in N$ 2p.

Inlocuind in (1) se obtin relatiile :

$x_{n+1}=\frac{a}{2}-\frac{1}{2}x_n$

$y_{n+1}=\frac{a}{2}-\frac{1}{2}y_n$ $\quad\forall n\in N$

$z_{n+1}=\frac{a}{2}-\frac{1}{2}z_n$

Pentru sirul x_n se obtine $x_n=\frac{a}{2}[1-(-\frac{1}{2})^n]+(-\frac{1}{2})^n x_0$ 1p.

Atunci $\lim_{n\to\infty} a_n=\lim_{n\to\infty} tg^2 x_n= tg^2\frac{x_0+y_0+z_0}{3}\Rightarrow\lim_{n\to\infty} a_n=tg^2\,\frac{arctg\sqrt{a_0}+arctg\sqrt{b_0}+arctg\sqrt{c_0}}{3}$

Analog pentru y_n si z_n. 1p.